Mathematical Finance
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Assignment 2—solutions

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

Approximating stopping times

Let τ be an $\mathbb{F}$–optional time. Define for any integer $n \in \mathbb{N}^\star$ the following random times, for any $\omega \in \Omega$

$$\tau_n(\omega) := \begin{cases} \tau(\omega), \text{ if } \tau(\omega) = +\infty, \\ \displaystyle\sum_{k=1}^{+\infty} \frac{k}{2^n} \mathbf{1}_{\{(k-1)/2^n \leq \tau(\omega) < k/2^n\}}. \end{cases}$$

Show that $(\tau_n)_{n\in\mathbb{N}^\star}$ is a non-increasing sequence of $\mathbb{F}$–stopping times, which converges to τ, and such that for any set $A \in \mathcal{F}_{\tau+}$, we have $A \cap \{\tau_n = k/2^n\} \in \mathcal{F}_{k/2^n}$, for any integers $(n, k) \in \big(\mathbb{N}^\star\big)^2$.

Fix some $n \in \mathbb{N}^\star$. For any $t \geq 0$, there exists a unique positive integer k_o such that $(k_o - 1)/2^n \leq t < k_o/2^n$. We thus have

$$\{\tau_n \leq t\} = \bigcup_{k=1}^{+\infty} \{k/2^n \leq t\} \cap \{(k-1)/2^n \leq \tau < k/2^n\} = \bigcup_{k=1}^{k_o-1} \{(k-1)/2^n \leq \tau < k/2^n\} \in \mathcal{F}_{(k_0-1)/2^n} \subset \mathcal{F}_t,$$

since τ is an $\mathbb{F}$–optional time. This implies that τ_n is an $\mathbb{F}$–stopping time.

Next, we have for any $\omega \in \Omega$ and any $n \in \mathbb{N}^\star$

$$\tau_{n+1}(\omega) = \tau_n(\omega) = +\infty, \text{ if } \tau(\omega) = +\infty, \text{ and } \tau_{n+1}(\omega) = \frac{\lfloor 2^{n+1}\tau(\omega)\rfloor + 1}{2^{n+1}} \leq \frac{\lfloor 2^n\tau(\omega)\rfloor + 1}{2^n} = \tau_n(\omega), \text{ if } \tau(\omega) < +\infty.$$

Indeed

$$\frac{\lfloor 2^n\tau(\omega)\rfloor + 1}{2^n} - \frac{\lfloor 2^{n+1}\tau(\omega)\rfloor + 1}{2^{n+1}} = \frac{2\lfloor 2^n\tau(\omega)\rfloor + 1 - \lfloor 2^{n+1}\tau(\omega)\rfloor}{2^{n+1}} \geq \frac{2^{n+1}\tau(\omega) - 1 - 2^{n+1}\tau(\omega)}{2^{n+1}} = -\frac{1}{2^{n+1}}.$$

However, since the left-hand side is an integer divided by 2^{n+1}, the above inequality actually proves that it is non-positive, which gives us the required monotonicity. Next, notice that for any $\omega \in \Omega$ and any $n \in \mathbb{N}^\star$, we either have

$$\tau_n(\omega) = \tau(\omega) = +\infty, \text{ or } \tau(\omega) < \tau_n(\omega) \leq \tau(\omega) + \frac{1}{2^n}.$$

This implies immediately that $\tau_n(\omega)$ converges to $\tau(\omega)$ as n goes to infinity. As for the final result, let us fix some $A \in \mathcal{F}_{\tau+}$. We have that for any integers k and n greater than 1

$$A \cap \{\tau_n = k/2^n\} = A \cap \{\tau < k/2^n\} \cap \{\tau \geq (k-1)/2^n\} \in \mathcal{F}_{k/2^n},$$

by definition of $\mathcal{F}_{\tau+}$.

Local boundedness and filtrations

Let U be a non-negative and unbounded random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$X_t := U\mathbf{1}_{\{t>1\}},\ t \geq 0,$$

and let $\mathbb{F} := \mathbb{F}^X$.

1) Show that $\mathbb{F}$ is not right-continuous and that a random time τ is an $\mathbb{F}$–stopping time if and only if τ is a deterministic number in $[0,1]$, or $\tau = f(U)$ for some Borel-measurable map $f : \mathbb{R} \longrightarrow (1,+\infty)$.

X is obviously càglàd, and it is clear that for any $t \in [0,1]$, $\mathcal{F}_t = \{\emptyset, \Omega\}$, while $\mathcal{F}_t = \sigma(U)$ for $t > 1$, which shows that right-continuity of $\mathbb{F}$ fails at $t = 1$. Next, the random times mentioned in the question are clearly $\mathbb{F}$–stopping times, as this is obvious in the deterministic case, and in the other case, we have for any $f : \mathbb{R} \longrightarrow (1,+\infty)$ and for any $t \geq 0$

$$\{f(U) \leq t\} = \begin{cases} \emptyset, \text{ if } t \in [0,1] \\ \{f(U) \leq t\} \in \mathcal{F}_{1+} \subset \mathcal{F}_t, \text{ if } t > 1. \end{cases}$$

Conversely, if τ is an $\mathbb{F}$–stopping time, we have $\{\tau \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. In particular, for $t \in [0,1]$ we must have $\{\tau \leq t\}$ equal to either $\emptyset$ or Ω, meaning that either τ is a deterministic constant in $[0,1]$ or $\tau > 1$. Next, for $t > 1$, we must have $\{\tau \leq t\} \in \sigma(U)$, and the $\sigma(U)$-measurable random variables are exactly given by $f(U)$ for some Borel-measurable map f, which must then be strictly above 1.

2) Show that X is not $(\mathbb{F},\mathbb{P})$–locally bounded.

If there existed a sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$–stopping times converging to $+\infty$ such that for any $n \in \mathbb{N}$, $\mathbf{1}_{\{\tau_n>0\}}X^{\tau_n}$ is bounded. Then, for $n \in \mathbb{N}$ sufficiently large, we must have $\tau_n > 1$, and we can therefore assume without loss of generality that there is a sequence $(f^n)_{n\in\mathbb{N}}$ of Borel-measurable maps from $\mathbb{R}$ to $(1,+\infty)$ such that $\tau_n = f^n(U)$, $n \in \mathbb{N}$. Notice then that

$$X_t^{\tau_n} = X_{f^n(U)\wedge t} = \begin{cases} 0, \text{ if } t \in [0,1], \\ U, \text{ if } t > 1, \end{cases}$$

and therefore X^{τ_n} cannot be $\mathbb{P}$–a.s. bounded.

Émery topology and change of measure

Show that the $(\mathbb{F},\mathbb{P})$–Émery topology is invariant under an equivalent change of measure.

It is clearly enough to show that if $(X^n)_{n\in\mathbb{N}}$ is a sequence converging to 0 for the $(\mathbb{F},\mathbb{P})$–Émery topology, then it also converges to 0 for the $(\mathbb{F},\mathbb{Q})$–Émery topology, for any equivalent measure $\mathbb{Q}$. Fix such a sequence and let $\mathbb{Q}$ be such an equivalent measure with Radon–Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} =: Z$. This means that for any every sequence $(\xi^n)_{n\in\mathbb{N}}$ of simple $\mathbb{F}$-predictable processes bounded by 1, and every $t \geq 0$, the following convergence holds in $\mathbb{P}$-probability

$$\xi_0^n X_0^n + \int_0^t \xi_s^n \mathrm{d}X_s^n \underset{n\to+\infty}{\longrightarrow} 0.$$

To prove that $(X^n)_{n\in\mathbb{N}}$ also converges for the $(\mathbb{F},\mathbb{Q})$–Émery topology, it therefore suffices to prove that convergence in $\mathbb{P}$-probability is equivalent to convergence in $\mathbb{Q}$-probability. Consider thus some sequence $(Y^n)_{n\in\mathbb{N}}$ converging in $\mathbb{P}$-probability to Y, and fix two positive constants ε and ε'. $\mathbb{P}$ and $\mathbb{Q}$ being equivalent, for any $A \in \mathcal{F}$, we can alway find some $\delta > 0$ such that if $\mathbb{P}[A] \leq \delta$, then $\mathbb{Q}[A] \leq \varepsilon'$. Now take $A := \{|Y^n - Y| > \varepsilon\}$. We know that for n large enough, we have $\mathbb{P}[|Y^n - Y| > \varepsilon] \leq \delta$, so that $\mathbb{Q}[|Y^n - Y| > \varepsilon] \leq \varepsilon'$ which proves the result by arbitrariness of ε and ε'.

Completeness and the Émery topology

We let $\mathcal{S}_b(\mathbb{F},\mathbb{P})$ be the space of bounded, simple $\mathbb{F}$-predictable processes, and we assume that $\mathbb{F}$ satisfies the usual conditions.

1) Show that the space of càdlàg and $\mathbb{F}$-adapted processes, as well as the space of càglàd and $\mathbb{F}$-adapted processes are complete under the $(\mathbb{F},\mathbb{P})$–Émery topology (you may use here the result of Proposition 4.6.5 and Theorem 4.6.2 from the lecture notes).

Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of càdlàg (resp. càglàd), $\mathbb{F}$-adapted processes, which is Cauchy under the $(\mathbb{F},\mathbb{P})$–Émery topology. By Proposition 4.6.5, this is also a Cauchy sequence for the $\mathbb{P}$–ucp convergence, so it has a $\mathbb{P}$–ucp limit X by Theorem 4.6.2, which is also càdlàg (resp. càglàd) and $\mathbb{F}$-adapted.

Then, since $X_t^n \longrightarrow_{n\to+\infty} X_t$ in $\mathbb{P}$-probability for each $t\geq 0$, it follows immediately that

$$\int_0^t \xi_s \mathrm{d}X_s^n \longrightarrow_{n\to+\infty} \int_0^t \xi_s \mathrm{d}X_s, \text{ in } \mathbb{P}\text{-probability},$$

for all simple $\mathbb{F}$-predictable processes ξ which are bounded by 1. Hence, setting $Y^n := X^n - X$, $n\in\mathbb{N}$, we deduce that for any such simple $\mathbb{F}$-predictable process ξ, we have

$$\mathbb{E}^{\mathbb{P}}\left[\left|\xi_0 Y_0^n + \int_0^t \xi_s \mathrm{d}Y_s^n\right| \wedge 1\right] = \lim_{m\longrightarrow+\infty} \mathbb{E}^{\mathbb{P}}\left[\left|\xi_0(X_0^n - X_0^m) + \int_0^t \xi_s \mathrm{d}(X^n - X^m)_s\right| \wedge 1\right] \leq \limsup_{m\to+\infty} d_{\mathrm{em},t}(X^n - X^m).$$

Taking the supremum over all ξ gives

$$\limsup_{n\to+\infty} d_{\mathrm{em},t}(X^n - X) \leq \limsup_{n\to+\infty}\limsup_{m\to+\infty} d_{\mathrm{em},t}(X^n - X^m) = 0,$$

since $(X^n)_{n\in\mathbb{N}}$ is Cauchy.

2) Show that if X is càdlàg process, the following are equivalent

(i) the map J_X from $\mathcal{S}_b(\mathbb{F},\mathbb{P})$ to the set of càdlàg and $\mathbb{F}$-adapted processes defined by

$$J_X(\xi) := \int_0^\cdot \xi_s \mathrm{d}X_s,$$

is continuous with respect to $\mathbb{P}$–ucp convergence on both spaces;

(ii) for every $t\in[0,+\infty)$, the mapping I_{X^t} from $\mathcal{S}_b(\mathbb{F},\mathbb{P})$ to $\mathbb{L}^0(\mathbb{R},\mathcal{F})$ with $I_{X^t}(\xi) := J_X(\xi)_t$, is continuous with respect to the uniform norm metric on $\mathcal{S}_b(\mathbb{F},\mathbb{P})$ and convergence in $\mathbb{P}$-probability on $\mathbb{L}^0(\mathbb{R},\mathcal{F})$.

Since J_X and $I_{X^t}, t\in[0,\infty)$ are linear mappings between topological vector spaces, it suffices to establish the continuity at the origin of $\mathcal{S}_b(\mathbb{F},\mathbb{P})$. Let X satisfy (i) and let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}_b(\mathbb{F},\mathbb{P})$ which converges uniformly to 0. This of course implies the weaker convergence $\xi^n \longrightarrow_{n\to+\infty} 0$ in the $\mathbb{P}$–ucp topology. Since X satisfies (i), $J_X(\xi^n)\longrightarrow_{n\to+\infty} 0$ for the $\mathbb{P}$–ucp topology. In particular, for any $t\in[0,\infty)$ we have $I_{X^t}(\xi^n)\longrightarrow 0$ in $\mathbb{P}$-probability, as claimed.

For the converse, let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}_b(\mathbb{F},\mathbb{P})$ converging $\mathbb{P}$–ucp to 0, and take $c>0$. Fix a $t\geq 0$ and $\varepsilon>0$. By assumption, we can find $\delta>0$ such that for any $n\in\mathbb{N}$, $\|H\|_{\mathbb{L}^\infty(\mathbb{R},\mathcal{F},\mathbb{P})}\leq\delta$, then $\mathbb{P}\big[\big|\int_0^t \xi_s^n \mathrm{d}X_s\big| > c\big] < \varepsilon$. Now, for each $n\in\mathbb{N}$ define the $\mathbb{F}$–stopping times (recall that $\mathbb{F}$ satisfies the usual conditions)

$$\tau^n := \inf\big\{t\geq 0 : \xi_t^n > \delta\big\}, \sigma^n := \inf\left\{t\geq 0 : \left|\int_0^t \xi^n \mathbf{1}_{[0,\tau^n]}(s)\mathrm{d}X_s\right| > c\right\}.$$

Then, the following inequalities hold for any $t\geq 0$

$$\begin{aligned}
\mathbb{P}\left[\sup_{0\leq s\leq t}\left|\int_0^s \xi_u^n \mathrm{d}X_u\right| > c\right] &\leq \mathbb{P}\left[\sup_{0\leq s\leq t}\left|\int_0^s \xi_u^n \mathbf{1}_{[0,\tau^n]}(u)\mathrm{d}X_u\right| > c\right] + \mathbb{P}[\tau^n \leq t] \\
&\leq \mathbb{P}\left[\sup_{0\leq s\leq t}\left|\int_0^s \xi_u^n \mathbf{1}_{[0,\tau^n\wedge\sigma^n]}(u)\mathrm{d}X_u\right| > c\right] + \mathbb{P}[\tau^n \leq t] \\
&\leq \varepsilon + \mathbb{P}[\tau^n\leq t] \\
&\leq 2\varepsilon,
\end{aligned}$$

for large enough n. The first line follows, since $\xi^n = \xi^n \mathbf{1}_{[0,\tau^n]}$ up to time t on the event $\{\tau^n > t\}$ holds. The second line is clear by definition of σ^n. For the third line, note that $\left\| \int_0^t \xi_s^n \mathbf{1}_{[0,\tau^n]} \right\|_{\mathbb{L}^\infty(\mathbb{R},\mathcal{F},\mathbb{P})} \leq \delta$ by definition of τ_n and left-continuity of H^n, and therefore we can use the assumption on I_t to bound this probability. For the last line, note that $\{\tau^n \leq t\} = \{\sup_{s\in[0,t]} |\xi_s^n| > c\}$ has low probability for large n, thanks to $\mathbb{P}$–ucp convergence.

Putting everything together, we see that $\int_0^\cdot \xi_u^n \mathrm{d}X_u$ converges $\mathbb{P}$–ucp to 0, as desired.